V. It has been claimed that the equation (A) above includes all possible solutions of

$$x^2 + y^2 + z^2 = 0,$$

but the writer is not at present prepared to affirm or deny that claim.

(B) REMARKS BY J. W. YOUNG, Dartmouth College.

After reading the remarks by W. C. Eells, on Algebra Problem 409, in the April number of the MONTHLY, it occurred to me that the geometric formulation for the right-triangle problem given by Klein in his *Elementarmath. v. höheren Standp. aus,* Vol. I, should yield a general solution also for the rectangular parallelepiped case. The problem is to solve in integers the equation

$$a^2 + b^2 + c^2 = d^2 \quad (> 0). \tag{1}$$

Placing $x = \frac{a}{d}$, $y = \frac{b}{d}$, $z = \frac{c}{d}$ the equation becomes

$$x^2 + y^2 + z^2 = 1. \tag{2}$$

Equation (1) will be completely solved, if we find all *rational* points on the sphere (2). $(-1, 0, 0)$ is one such point. Any straight line through $(-1, 0, 0)$ and any other rational point will have the equations

$$\frac{x+1}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu}, \tag{3}$$

with λ, μ, ν integral and without a common factor; and conversely, every such line (3) will cut out a rational point, P, besides $(-1, 0, 0)$. Solving (2) and (3) we find

$$P = \left(\frac{\lambda^2 - \mu^2 - \nu^2}{\lambda^2 + \mu^2 + \nu^2}, \frac{2\mu\lambda}{\lambda^2 + \mu^2 + \nu^2}, \frac{2\nu\lambda}{\lambda^2 + \mu^2 + \nu^2}\right).$$

If D represents the H. C. F. of $\lambda^2 - \mu^2 - \nu^2$, $2\mu\lambda$, $2\nu\lambda$, and $\lambda^2 + \mu^2 + \nu^2$, then the *general solution* of (1) is given by

$$(a, b, c, d) = \left(\frac{k}{D}(\lambda^2 - \mu^2 - \nu^2), \frac{k}{D}\cdot 2\mu\lambda, \frac{k}{D}\cdot 2\nu\lambda, \frac{k}{D}(\lambda^2 + \mu^2 + \nu^2)\right), \tag{4}$$

where k is any integer.

Y. A. Le Besgue's solutions, to which reference has already been made by Mr. Eells, yield, for $\delta = 0$, the solutions above for $K = D$. If λ, μ, ν are, *e. g.*, relatively prime, no solutions are duplicated by (4). Moreover it is then easy to see that D is equal to the H. C. F. of λ and $\mu^2 + \nu^2$ or twice this H. C. F. The above method will clearly hold for the general equation $\sum_{i=1}^{n} x_1^2 = n^2$.

420. Proposed by ELBERT H. CLARKE, Purdue University.

Given the infinite series,

$$\frac{a}{r} + \frac{b}{r^2} + \frac{a+b}{r^3} + \frac{a+2b}{r^4} + \frac{2a+3b}{r^5} + \cdots,$$

in which a and b are any numbers, and where each numerator after the second is the sum of the two preceding numerators. To find the region of convergence and the sum of the series.

This problem is a generalization of one solved in the January (1914) number of the MONTHLY.

II. SOLUTION BY THE PROPOSER.

Let us designate the above numerators as follows:

$$U_1 = a;\; U_2 = b;\; U_3 = a + b;\; U_n = U_{n-1} + U_{n-2}.$$

Also let 1, 1, 2, 3, 5, 8 $\cdots u_n = u_{n-1} + u_{n-2}$ be a special sequence of the above type. Then by inspection, easily verified by induction, we have

$$U_n = au_{n-2} + bu_{n-1},\ n > 2. \tag{1}$$

And furthermore, the equation $\rho^n = \rho^{n-1} + \rho^{n-2}$ gives the two solutions

$$\rho_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \rho_2 = \frac{1-\sqrt{5}}{2}.$$

Then a sequence of powers of either ρ_1 or ρ_2 will be a sequence obeying the fundamental relation (1).

Therefore

$$\rho_1{}^n = \rho_1 u_{n-2} + \rho_1{}^2 u_{n-1},$$

$$\rho_2{}^n = \rho_2 u_{n-2} + \rho_2{}^2 u_{n-1}.$$

Since

$$\rho_1 + \rho_2 = 1;\ \rho_1 - \rho_2 = \sqrt{5};\ \rho_1{}^2 - \rho_2{}^2 = \sqrt{5}$$

we have

$$\frac{\rho_1{}^n - \rho_2{}^n}{\sqrt{5}} = u_{n-2} + u_{n-1} = u_n. \tag{2}$$

We may now combine (1) and (2) and obtain

$$U_n = a\frac{\rho_1{}^{n-2} - \rho_2{}^{n-2}}{\sqrt{5}} + b\frac{\rho_1{}^{n-1} - \rho_2{}^{n-1}}{\sqrt{5}}. \tag{3}$$

In order to find the region of convergence, we may use the ratio test:

$$\frac{U_{n+1}}{r^{n+1}} \div \frac{U_n}{r^n} \text{ reduces to } \frac{1}{r}\left[\frac{a + \dfrac{u_n}{u_{n-1}}b}{\dfrac{u_{n-2}}{u_{n-1}}a + b}\right].$$

From the fundamental relation $u_n = u_{n-1} + u_{n-2}$ we have

$$\frac{u_n}{u_{n-1}} = 1 + \frac{u_{n-2}}{u_{n-1}} \text{ and therefore } \lim_{n\to\infty}\frac{u_n}{u_{n-1}} = 1 + \lim_{n\to\infty}\frac{u_{n-2}}{u_{n-1}},$$

provided that these limits exist. But $\dfrac{u_n}{u_{n-1}}$ is the $(n-1)$st convergent of the simple nonterminating continued fraction

$$1 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\cdots,$$

and therefore it has a limit; call it K. But $\lim\limits_{n\to\infty} u_{n-2}/u_{n-1} = 1/\lim\limits_{n=\infty} u_n/u_{n-1}$, and so $k = 1 + (1/k)$; giving $k = (1 \pm \sqrt{5})/2$; but since all u's are positive, $k = (1+\sqrt{5})/2 = \rho$. Our test ratio now reduces, when the limit is taken, to ρ_1/r.

Hence, the series

$$\sum_{n=1}^{\infty}\frac{U_n}{r^n}$$

will converge if $|r| > \rho_1$ and will diverge if $|r| < \rho_1$; where $\rho_1 = (1+\sqrt{5})/2$. By actual substitution of ρ_1 for r in the series, it may be shown that the series will also diverge for $|r| = \rho_1$. $\therefore$ The region of convergence is $|r| > [(1+\sqrt{5})/2)]$. Now, by making use of (3) we may obtain

$$\sum_{n=1}^{\infty} \frac{U_n}{r^n} = \frac{1}{\sqrt{5}}\left(\frac{a+b\rho_1}{\rho_1(r-\rho_1)} - \frac{1}{\sqrt{5}}\frac{a+b\rho_2}{\rho_2(r-\rho_2)}\right).$$

The right-hand member now reduces to

$$\frac{b+ar-a}{r^2-r-1}$$

when ρ_1 is replaced by $(1+\sqrt{5})/2$ and ρ_2 by $(1-\sqrt{5})/2$.

Note.—We are publishing this solution for the reason that the previously published solution referred to did not consider the question of convergency correctly, and the proper investigation of this question was the Proposer's chief reason for proposing the problem. EDITORS.

ALGEBRA.

429. Proposed by C. N. SCHMALL, New York City.

It is given that d_1, d_2, d_3 are the greatest common divisors of y and z, z and x, x and y, respectively; also that m_1, m_2, m_3 are the least common multiples of the same pairs of members. If d and m are the greatest common divisor and least common multiple, respectively, of x, y, and z, show that

$$\frac{m}{d} = \left(\frac{m_1m_2m_3}{d_1d_2d_3}\right)^{\frac{1}{2}}.$$

SOLUTION BY FRANK IRWIN, University of California.

It is evident that we can get the least common multiple of two numbers by dividing their product by their greatest common divisor:

$$m_1 = \frac{yz}{d_1},\ m_2 = \frac{xz}{d_2},\ m_3 = \frac{xy}{d_3}.$$

Similarly with the three numbers x, y, z, if we divide their product by $d_1d_2d_3$, we should have their least common multiple, except that we have divided out d once too often:

$$m = \frac{xyz}{d_1d_2d_3}\cdot d.$$

We have then:

$$\left(\frac{m_1}{d_1}\cdot\frac{m_2}{d_2}\cdot\frac{m_3}{d_3}\right)^{\frac{1}{2}} = \left(\frac{yz}{d_1^2}\cdot\frac{zx}{d_2^2}\cdot\frac{xy}{d_3^2}\right)^{\frac{1}{2}} = \frac{xyz}{d_1d_2d_3} = \frac{m}{d}.$$

Also solved by A. H. HOLMES, ELMER SCHUYLER, G. W. HARTWELL, FRANK R. MORRIS, N. P. PANDYA, HERBERT N. CARLETON, PAUL CAPRON, J. A. CAPARO, and the PROPOSER.

GEOMETRY.

443. Proposed by C. N. SCHMALL, New York City.

A quadrilateral of any shape whatever is divided by a transversal into two quadrilaterals. The diagonals of the original figure and those of the two resulting (smaller) figures are then drawn. Show that their three points of intersection are collinear.

III. SOLUTION BY LAENAS G. WELD, Pullman, Ills.

To the triangle ABC draw the transversals MN, intersecting AB in M and AC in N, and QR intersecting AB in Q and AC in R. Then $BCRQ$, $BCNM$ and